Exploring the Cubic Formula

NW Iowa Math Teachers Circle

Mike Janssen, Ph.D.

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1 Introduction

By now we have all learned the quadratic formula, which gives solutions to the equation $ax^2 + bx + c = 0$ in terms of the coefficients a, b, c and standard arithmetic operations of addition, subtraction, multiplication, division, and the taking of radicals.

But what about higher-degree polynomials?

It turns out that there are formulas for cubic and quartic polynomials as well, and our goal today is to explore the cubic formula, its history, and some of the mathematical questions it raises.

2 A bit of history

2.1 Babylonians and Muslims

The solution to the quadratic was known by the Babylonians, over 5000 years ago, but it is closely associated with the work of the Islamic mathematicians, who were active in the Middle East around 800–1200 AD. One of those mathematicians, Omar Khayyam, was able to solve cubic equations by intersecting conic sections, but, as negative numbers were viewed as fictitious, he had 14 different versions of a cubic:

1. $x^3 = d$
2. $x^3 + cx = d$
3. $x^3 + d = cx$
4. $x^3 = cx + d$
5. $x^3 + bx^2 = d$
6. $x^3 + d = bx^2$
7. $x^3 = bx^2 + d$
8. $x^3 + bx^2 + cx = d$
9. $x^3 + bx^2 + d = cx$
10. $x^3 + cx + d = bx^2$
11. $x^3 = bx^2 + cx + d$
12. $x^3 + bx^2 = cx + d$
13. $x^3 + cx = bx^2 + d$
14. $x^3 + d = bx^2 + cx$

(In this case, $b, c, d > 0$.)

Khayyam lacked a general approach to solving cubics, and this remained the case through the early 1500s.

2.2 The Italians

By 1500, it was known that one could reduce the general cubic

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

to the so-called *depressed* cubic

$$u^3 + eu + f = 0 \tag{2}$$

So, if one could solve the depressed cubic, one could reverse-engineer a solution to a general cubic.

Exercise

Consider the cubic $x^3 + 6x^2 - 25$. Use the substitution $x = u - 2$ to transform it into a depressed cubic in the variable u. (Credit: Cal Jongsma)
(If one divides the general cubic Equation 1 by a and substitutes $x = u - b/(3a)$, the square term is eliminated.)

2.2.1 1535: Fiore vs Tartaglia

In modern academia, a researcher is rewarded for their discoveries: prestige, grant funds, and prominent speaking slots at major conferences can all follow from a major breakthrough. This was not the case in Renaissance Italy. University jobs were temporary, and one needed to continually prove oneself worthy by winning public competitions.

In the early 1500s, the mathematician Scipione del Ferro discovered how to solve cubics of the form $x^3 + cx = d$. He passed the solution onto his student, Antonia Maria Fiore. Around the same time, Niccolo Fontana, known as Tartaglia, discovered how to solve $x^3 + bx^2 = d$.
Fiore heard of Tartaglia's boasts, and challenged him to a competition in 1535. They each posed 30 problems for the other to solve, most (all?) of which boiled down to solving a cubic. For example:

One of del Ferro's problems

A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is this profit?

This problem boils down to solving $x^3 + x = 500$. Tartaglia worked to learn the solution to cubics of that form, while del Ferro was unable to solve the cubics that Tartaglia posed. Tartaglia won the competition, and with it, money and a job as a university lecturer.

2.2.2 Gerolamo Cardano (1501–1576)

Gerolamo Cardano was another lecturer in mathematics, as well as a sought-after physician. Still, he had his troubles; he gambled, he struggled with misbehaving sons, he was jailed during the Inquisition and more. Nonetheless, decades later, Gottfried Leibniz wrote, “Cardano was a great man with all his faults; without them, he would have been incomparable.” His collected works fill 7,000 pages and include the first serious investigations of probability theory.

Cardano wanted Tartaglia’s solution for a new arithmetic text he was writing. Tartaglia resisted for a while, but finally came to Milan in 1539 to share the secret with Cardano, who in turn pledged the following:

> I swear to you, by God’s holy Gospels, and as a true man of honor, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them.

Tartaglia gave his solution to $x^3 + cx = d$ in a poem:

> When the cube and its things near
>
> Add to a new number, discrete,
>
> Determine two new numbers different
>
> By that one; this feat
>
> Will be kept as a rule
>
> Their product always equal, the same,
>
> To the cube of a third
>
> Of the number of things named.
>
> Then, generally speaking,
>
> The remaining amount
>
> Of the cube roots of subtracted
>
> Will be your desired count.

3 Cardano's Formula

We begin with the depressed cubic $u^3 + eu + f = 0$ (Equation 2). Observe that the binomial formula gives

$$(s+t)^3 = s^3 + 3s^2t + 3st^2 + t^3, \tag{3}$$

Exercise

Rewrite Equation 3 as a cubic in the variable $s+t$. [Hint: the constant term will be $-s^3 - t^3$.]

The work in the previous exercises means that $u = s + t$ is a solution of $u^3 + eu + f = 0$ if s and t are chosen to solve the system

$$\begin{aligned} -3st &= e \\ s^3 + t^3 &= -f. \end{aligned}$$

Exercise

Let's finish the derivation of the cubic formula.

1. Solve the first equation in the system above for s.

2. Substitute the solution for s into the second equation; rewrite as a *quadratic* equation in the variable t^3.

3. Use the quadratic formula to solve for t^3, and then take cube roots to solve for t.

4. Finally, show that

$$u = -\frac{e}{3\sqrt[3]{\frac{-f}{2} + \frac{\sqrt{f^2+4e^3/27}}{2}}} + \sqrt[3]{\frac{-f}{2} + \frac{\sqrt{f^2+4e^3/27}}{2}} \tag{4}$$

Let's put Equation 4 to good use and solve a cubic that Cardano himself thought about.

Exercise

In this exercise we'll consider the cubic $x^3 - 15x - 4 = 0$.

1. Using the general depressed cubic (Equation 2), identify e and f.

2. Now use Equation 4 to write the solution u—but clear the denominator of all cube roots. Simplify as much as possible to obtain the solution $u = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$.

In solving this equation, Cardano noticed that the formula could produce square roots of negative numbers while nonetheless having positive real solutions (we'll see it shortly!). He asked Tartaglia about it, but Tartaglia seemed to have no answer, and suggested that perhaps Cardano had not properly understood how to solve such problems. Cardano later wrote that working with square roots of negative numbers caused him "mental tortures".

It fell to the next generation of Italian mathematicians, particularly Rafael Bombelli, to describe the arithmetic of what we now call *complex numbers*. In order to better understand the solution we just found, let's take a detour into this larger realm.

4 Complex Numbers

4.1 Forms of a complex number

With the benefit of 400+ years of mathematical development after Bombelli, let's explore complex numbers to understand just what $\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$ really is.

Definition 4.1. A *complex number* is an expression of the form $a+bi$, where $a,b\in\mathbb{R}$ and $i^2=-1$. The set of complex numbers is denoted by $\mathbb{C}$. The *real part* of the complex number, denoted $\Re(a+bi)$, is the real number a. The *imaginary part* of the complex number, denoted $\Im(a+bi)$, is the real number b.

Since a complex number is identified by two bits of information (its real and imaginary parts), we can visualize the complex numbers in a plane, as shown below.

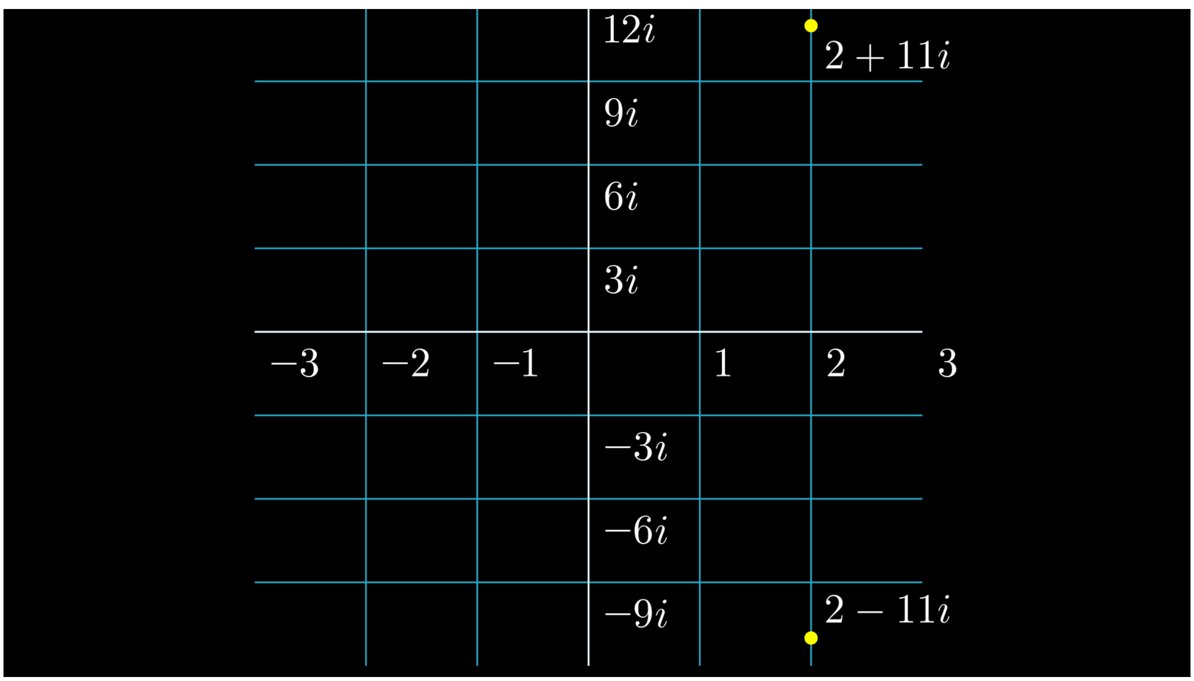


Figure 1: Points in the complex plane.

Exercise

Let's try to understand what $\sqrt[3]{2+11i}$ is. If $\sqrt[3]{2+11i} = a + bi$, we may cube both sides to find $(a+bi)^3 = 2 + 11i$.

1. Expand the left hand side, and set the real part equal to 2 and the imaginary part equal to 11.

2. Verify that $a = 2$ and $b = 1$ is a solution to the two equations you found.

3. Do something similar for $\sqrt[3]{2-11i}$ and verify that $a = 2, b = -1$ is a solution.

4. Thus, what number is $\sqrt[3]{2+11i} + \sqrt[3]{2-11i}$?

This *ad hoc* approach is a little unsatisfying. Thankfully, there is another way to take the cube root of $2 + 11i$ using the *polar form* of a complex number.

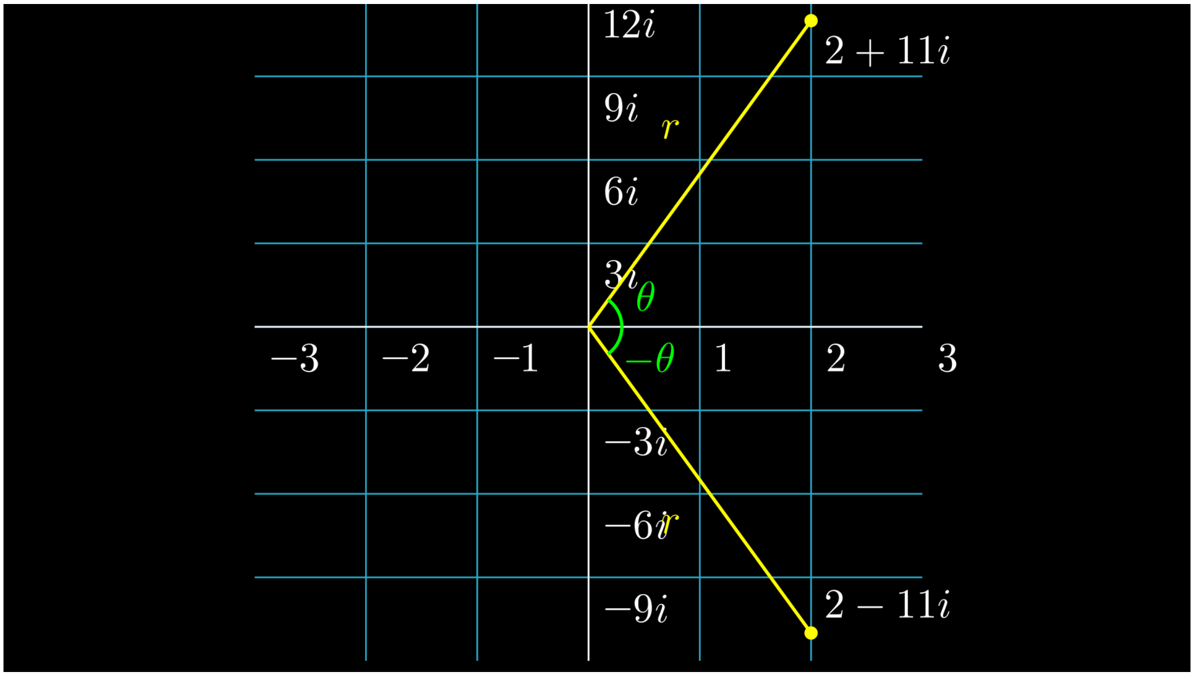


Figure 2: Motivating the polar form.

Using elementary trigonometry, we see that for some r and θ,

$$a + bi = r(\cos\theta + i\sin\theta). \tag{5}$$

Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, allows us to rewrite Equation 5 as

$$a + bi = re^{i\theta}. \tag{6}$$

Exercise

Let's write $2 + 11i$ and its cube root in polar form.

1. Calculate r.

2. Give an expression for θ involving the arctangent function, and write $2+11i$ using both the trigonometric (Equation 5) and exponential (Equation 6) polar forms of a complex number.

3. Suppose the angle for the polar form of $\sqrt[3]{2+11i}$ is called α. How must α and θ relate?

4. Finally, write $\sqrt[3]{2+11i}$ in polar form.

What we found in the previous exercise is that $\alpha = \theta/3$. We will use the triple angle formula for tangent to evaluate $\sqrt[3]{2+11i}$. Recall that

$$\tan 3\alpha = \frac{\tan\alpha(3-\tan^2\alpha)}{1-3\tan^2\alpha}. \tag{7}$$

Exercise

Let's try to understand what $\sqrt[3]{2+11i}$ is, this time using the polar form of a complex number, as well as Equation 7.

1. First, calculate $\tan 3\alpha = \tan\theta$.

2. Let $t = \tan\alpha$, and rewrite Equation 7 using the value you found in #1.

3. Clear denominators and rewrite your equation from #2 as a cubic.

4. Observe that $t = 1/2$ is a solution to the cubic you found; thus, $\tan(\theta/3) = 1/2$ is a solution, and $\alpha = \arctan(1/2)$.

5. Calculate $\cos(\alpha)$ and $\sin(\alpha)$, and use these values to show that $\sqrt[3]{2+11i} = 2+i$.

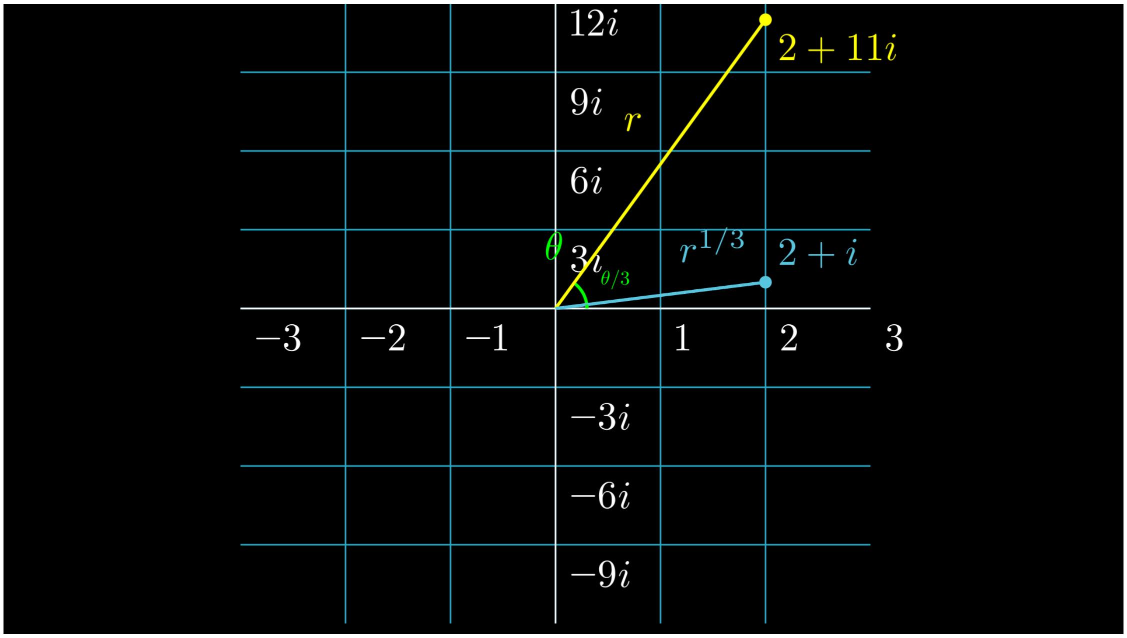


Figure 3: Visualizing the cube root of $2 + 11i$.

4.2 Roots of Unity: Finding all solutions of the cubic

So, Cardano's formula can produce some surprising identities, like $\sqrt[3]{2+11i} + \sqrt[3]{2-11i} = 4$. "But wait!" you exclaim. "shouldn't cubic equations have *three* complex solutions (with multiplicity)? Cardano's formula produces only one!". As presented here, that is true. There are two ways out of this box.

One is to use the *Factor Theorem*, which states that if $p(x)$ has a root α, then $x - \alpha | p(x)$. We can therefore divide $x^3 - 15x - 4$ by $x - 4$ to produce the quadratic $x^2 + 4x + 1$ and apply the quadratic formula.

A second, more interesting, way out of the box is to use *roots of unity*.

Definition 4.2. Given a natural number $n \geq 1$, a complex number z is called an *nth root of unity* if $z^n = 1$. An nth root of unity z_0 is called *primitive* if it is not a root of unity for any $m < n$.

Exercise

Let's explore some elementary properties of roots of unity.

1. Explain why 1 and -1 are roots of unity for all even n. Are either of them primitive roots of unity for some n?

2. For $k = 0, 1, 2, \ldots, n-1$, explain why $e^{2k\pi i/n}$ is a root of unity.

3. How many **distinct** nth roots of unity are there? (Hint: if $re^{i\theta}$ is a root of unity, what must r be?)

4. Use Equation 5 to write a formula for the nth roots of unity using the sine and cosine functions. Use this formula to fully describe the third roots of unity.

5. Suppose $\sqrt[n]{\alpha}$ is a solution of $z^n = \alpha$ and that ζ_n is an nth root of unity. Explain why $\zeta_n \sqrt[n]{\alpha}$ is also a solution of $z^n = \alpha$.

6. **Bonus challenge**: for a given n, how many *primitive* nth roots of unity are there?

Primitive roots of unity are nice because they *generate* the other roots of unity.

Theorem 4.1. *Let ζ_0 be a primitive nth root of unity. Then the nth roots of unity are given by* $1, \zeta_0, \zeta_0^2, \zeta_0^3, \ldots, \zeta_0^{n-1}$.

It turns out that while 1 is always a root of unity, it is not a primitive root of unity for any $n > 1$, but the other two cube roots of unity are both primitive. Using the formula you found for expressing roots of unity in terms of sine and cosine, we find the third roots of unity

$$\omega_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{8}$$

and

$$\omega_2 = \cos\left(\frac{2\cdot 2\pi}{3}\right) + i\sin\left(\frac{2\cdot 2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \tag{9}$$

Exercise

Confirm that $\omega_1^2 = \omega_2$.

4.2.1 This is nice, but why do we care?

Well, when we were deriving Cardano's formula above, we got to the point where we had

$$t^3 = \frac{-f}{2} + \frac{\sqrt{f^2 + 4e^3/27}}{2}.$$

But just as there are two *square* roots of a real number, there are *three cube roots of a nonzero real number*[1] c: $c^{1/3}, \omega_1 c^{1/3}$, and $\omega_1^2 c^{1/3}$. Each choice of a third root of unity will produce a different solution to the cubic, and so we collect all three, as the Fundamental Theorem of Algebra promises.

[1] And, indeed, there are n complex n-th roots of a nonzero real number.

Exercise: Finding all three roots

Let's return again to $x^3 - 15x - 4 = 0$. We saw that $x = s + t$, where $t = \sqrt[3]{\frac{-f}{2} + \frac{\sqrt{f^2+4e^3/27}}{2}} = 2 + i$ and $s = -e/3t$.

1. Using the notation of Equation 9, justify that $[(2+i)\omega_1]^3 = 2 + 11i$. (You don't need to work it all out by hand if you don't want to!)

2. Let $t_1 = (2+i)\omega_1$ and $s_1 = -e/3t_1$. Show that $s_1 = (2-i)\omega_2$.

3. Use Wolfram|Alpha or similar to verify that $x_1 = s_1 + t_1$ is a solution to $x^3 - 15x - 4 = 0$.

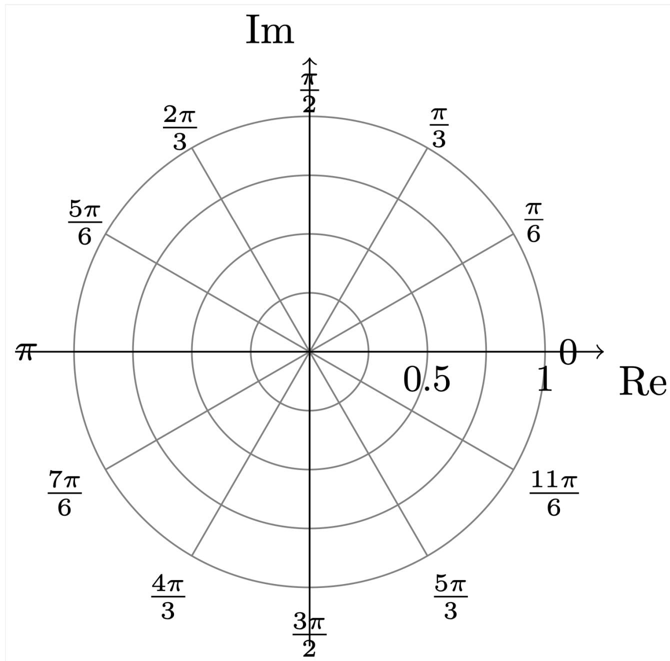
4. Conjecture the third solution, $x_2 = t_2 + s_2$. Confirm your conjecture.

4.2.2 Visualizing the roots of unity

The roots of unity have a pleasing visual form as well.

Exercise

1. Plot the n-th roots of unity for $n = 2, 3, 4$ below.



2. How might you plot the fifth roots of unity?

4.3 If time!

Solve the cubic $x^3 + 6x = 20$.

5 Historical coda

Eventually, Cardano and his student, Lodovico Ferrari, worked out the full solution to the cubic, but did not have permission from Tartaglia to publish. Cardano evidently heard that del Ferro found the solution before Tartaglia, and went to Bologna to check del Ferro's notes. Once he'd confirmed that Tartaglia was not, in fact, the first to solve the cubic, he felt released from his oath, and published the full solution to the cubic and quartic in the *Ars Magna*: "Written in five years, may it last as many thousands!"

According to Quanta Magazine:

> Tartaglia was livid, even though Cardano acknowledged his work in the book. Tartaglia accused Cardano of theft and of breaking a sacred vow. Cardano left the rebukes to his loyal attack dog, Ferrari. The acrimonious back-and-forth, in the form of public pamphlets, continued for many months, leading to a mathematical duel between Tartaglia and Ferrari and eventually a public debate in Ferrari's hometown, Milan. Tartaglia would much rather have battled the esteemed Cardano, but Cardano refused. Details are scarce, but the debate went terribly for Tartaglia, especially with the raucous hometown crowd. The next day, when it was time to continue the debate, Tartaglia was nowhere to be found—he'd left Milan.
>
> Ferrari was flooded with job offers, and Tartaglia's reputation was ruined. Despite many notable accomplishments beyond those related to the cubic, Tartaglia died penniless and largely unknown, whereas Cardano achieved everlasting fame. Many argue that the publication of Ars Magna marked the beginning of modern mathematics.