Research Statement

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Undergraduate Research

One of the great joys of my career thus far has been the opportunity to work with undergraduate students on meaningful mathematical research questions. In addition to two honors independent study projects, I've supervised or co-supervised five summer research projects, each for 1-2 students. These projects have led to multiple undergraduate conference presentations, a publication in the *Journal of Algebra and its Applications*, a publication in the *Rose-Hulman Undergraduate Math Journal*, and another submission currently in preparation. As a first-generation college student myself, I had no idea what it meant to do research in any field, much less mathematics, and so I relish the opportunity to open mathematical research up to students of all backgrounds.

While my own primary research interests, as well as the first project I supervised (described in the next section), are in commutative algebra and required substantial prerequisite knowledge, I've tailored subsequent projects to require fewer prerequisite courses. I've thus been able to encourage a broader group of students to consider applying, including students finishing their first year of the math major[1].

I also take care to structure my projects so that students always have multiple things to be working on across a variety of energy levels: some "low-hanging fruit" sorts of questions, deeper/bigger questions, as well as books to read and regular updates to a working draft of a paper. This diminishes the likelihood that a student will feel completely stuck and not feel like they have anything productive they can work on.

At the suggestion of a colleague, the 2019 edition of our summer research group focused on questions raised by [1,2]. In their work, they defined a directed graph associated to a finite (commutative) ring R: given $a, b \in R$, the vertex (a, b) points to $(a + b, a \cdot b)$. This encodes the ring's Cayley tables as a single discrete structure. The authors then characterize many familiar graph-theoretic concepts, such as components, cycles, in-degree, etc., in terms of the structure of the ring.

In 2020, my student wondered what these graphs would look like under different operations, so we adapted min-plus tropical addition to a finite structure and studied the resulting graphs [3]. Again, we obtained results characterizing various aspects of the graph in terms of properties of the structure (a "near-semiring"). We further explored this structure in the summer of 2021. In summer of 2022, my students and I explored *the game of cycles*, a finite combinatorial game introduced by Francis Su in [4] and further explored in [5]. We solved the game for a certain class of graphs, and one student developed a computational means of solving the game given any finite game board.

[1] In fact, one of my summer 2021 students has said that she never considered summer research as a possibility until I encouraged her to apply. She completed a second REU in summer 2022 at another institution.

Commutative Algebra

Introduction to the Containment Problem

Given an ideal I in a (commutative) ring R, there are lots of arithmetic operations one can perform on the ideal. Students in a first course in ring theory learn about sums and products, and those products can be iterated to obtain powers I^r for all $r \in \mathbb{N}$.

A related notion is that of a **symbolic power**.

Definition 1. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over the field k and $0 \neq I \subsetneq R$ a homogeneous ideal. The m-th symbolic power of I, denoted $I^{(m)}$, is the ideal*

$$I^{(m)} = \cap_P (I^m R_P \cap R),$$

where R_P denotes the localization of R at the prime ideal P, and the intersection is taken over all associated primes P of I.

The symbolic power typically carries extra information about objects related to I. For instance, if I is defined by points $p_1, p_2, \ldots p_N \in \mathbb{P}^2_k$, the ideal $I^{(m)} \subseteq k[x_0, x_1, x_2]$ is generated by all homogeneous polynomials vanishing to order at least m at the points.

Unfortunately, the generators of $I^{(m)}$ are typically complicated to write down, and no general procedure exists for finding them. A comparison to the relatively straightforward "ordinary power" I^r is natural.

Thus, over the past couple of decades, there has been a great deal of interest in the *ideal containment problem*: given a non-trivial homogeneous ideal I of a polynomial ring $R = k[x_0, x_1, \ldots, x_n]$ over a field k, the problem is to determine all positive integer pairs m, r such that $I^{(m)} \subseteq I^r$. An asymptotic version of the ideal containment problem is to calculate an invariant known as the **resurgence** of I:

$$\rho(I) = \sup\{m/r \mid I^{(m)} \not\subseteq I^r\}.$$

Calculating $\rho(I)$ is difficult in general, so many have taken to bounding it in terms of other constants. One such other constant is Waldschmidt's constant, $\hat{\alpha}(I)$:

$$\hat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m},$$

where $\alpha(J)$ is the degree of a nonzero polynomial of least degree in the nontrivial ideal J. Bocci and Harbourne, for instance, show that $\alpha(I)/\hat{\alpha}(I) \leq \rho(I)$ in [6]. In [7], two undergraduates and I explored the containment problem for a class of **edge ideals**.

Edge Ideals

We begin with a definition.

Definition 2. *Let $G = (V, E)$ be a (simple) graph with $V = \{x_1, x_2, \dots, x_n\}$. Then the edge ideal $I(G) \subseteq k[x_1, \dots, x_n]$ of G is the ideal generated by products of pairs of variables corresponding to edges in G:*

$$I(G) = \langle \{x_i x_j \mid \{x_i, x_j\} \in E\} \rangle.$$

The edge ideal encodes the structure of the graph in an algebraic setting, and has many pleasing properties relating it to the underlying graph! As an example, we recall the definition of a (minimal) vertex cover.

Definition 3. *Let $G = (V, E)$ be a simple graph with $V = \{x_1, x_2, \dots, x_n\}$. A vertex cover K of G is a set $K \subseteq V$ such that for all $e \in E$, $e \cap K \neq \emptyset$. A vertex cover K is said to be minimal if no proper subset of K is a vertex cover of G.*

A first helpful result establishing a link between finite simple graphs G and their edge ideals is the following (see Corollary 1.35 of [8]).

Theorem 1. *Let $K_1, K_2, \dots, K_t$ be the minimal vertex covers of a finite simple graph $G = (V, E)$ with vertices $V = \{x_1, x_2, \dots, x_n\}$. Set $\langle W_i \rangle = \langle x_j | x_j \in W_i \rangle$. Then*

$$I(G) = \langle W_1 \rangle \cap \cdots \cap \langle W_t \rangle.$$

My own recent work has built on these results and those of [8–11]. For instance, in Theorem 4.6 of [12], we relate $\hat{\alpha}(I(G))$ to well-known graph invariants via the following result, obtained via linear programming:

Theorem 2 (J–, et. al (2017))**.** *Suppose that $H = (V, E)$ is a hypergraph with a nontrivial edge and let $I = I(H)$. Then*

$$\hat{\alpha}(I) = \frac{\chi^*(H)}{\chi^*(H) - 1},$$

where $\chi^*(H)$ is the fractional chromatic number of H.

We are then able to relate $\hat{\alpha}(I(G))$ to properties of G for several classes of graphs.

Theorem 3 (J–, et. al (2017))**.** *Let G be a nonempty graph.*

(i) If $\chi(G) = \omega(G)$, then $\hat{\alpha}(I(G)) = \frac{\chi(G)}{\chi(G)-1}$.

(ii) If G is k-partite, then $\hat{\alpha}(I(G)) \geq \frac{k}{k-1}$. In particular, if G is a complete k-partite graph, then $\hat{\alpha}(I(G)) = \frac{k}{k-1}$.

(iii) *If G is bipartite, then* $\hat{\alpha}(I(G)) = 2$.

(iv) *If $G = C_{2n+1}$ is an odd cycle, then* $\hat{\alpha}(I(C_{2n+1})) = \frac{2n+1}{n+1}$.

(v) *If $G = C_{2n+1}^c$, then* $\hat{\alpha}(I(G)) = \frac{2n+1}{2n-1}$.

Future Work

A notion related to the containment problem is that of the **symbolic defect**, introduced in [13], which measures the failure of I^m to equal $I^{(m)}$ by considering the minimal number of generators required to generate $I^{(m)}$ from I^m. For many classes of graphs, the particular properties of their associated edge ideals, such as their symbolic defects, resurgences, and Waldschmidt constants, remain unexplored, and are largely accessible to well-prepared and motivated undergraduates. In [7], a paper I wrote with two Dordt undergraduates, we calculate the symbolic defect for edge ideals of odd cycles, in addition to solving the containment problem for these ideals.

Scholarly Teaching

Given my current context teaching 24+ credit hours/year, much of my scholarly output has had some synergy with my teaching. This has often meant writing and speaking on particular approaches to teaching my courses, but recently has included the production of open educational resources such as my rings-first, inquiry-oriented algebra text, *Rings with Inquiry*[2], and my in-progress liberal arts math text, *Explorations in Modern Math*[3]. I received a grant from the Iowa Private Academic Libraries (IPAL) for the production of the latter, and have been able to use some of those funds to involve two student assistants in the process. This work has been incredibly fun and immediately helpful for me in the courses I'm teaching. Such work is never truly done, so it will continue to be a feature of my scholarship in future years.

[2] ringswithinquiry.org
[3] emmath.org

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