Notes on Cardano's formula

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Begin with a general cubic

$$ax^3 + bx^2 + cx + d = 0.$$

By dividing by a and substituting $x = u - b/(3a)$, we obtain the depressed cubic

$$u^3 + eu + f = 0.$$

Observe that the binomial formula gives

$$(s+t)^3 = s^3 + 3s^2t + 3st^2 + t^3,$$

which we may rewrite as

$$(s+t)^3 - 3st(s+t) - s^3 - t^3 = 0.$$

This means that $u = s+t$ is a solution of $u^3 + eu + f = 0$ if s and t are chosen to solve the system

$$\begin{aligned} -3st &= e \\ s^3 + t^3 &= -f. \end{aligned}$$

We can solve the first equation to obtain $s = -e/(3t)$. If we substitute into the second and clear denominators, we obtain

$$(t^3)^2 + ft^3 - e^3/27 = 0,$$

which is quadratic in t^3. The Babylonians showed us how to find t^3; we can then take a cube root to get t, which in turn gives us s, and, finally, u. In fact, we find

$$t = \sqrt[3]{\frac{-f}{2} \pm \frac{\sqrt{f^2 + 4e^3/27}}{2}} \tag{1}$$

$$s = -e/3t = -\frac{e}{3\sqrt[3]{\frac{-f}{2} \pm \frac{\sqrt{f^2+4e^3/27}}{2}}}$$

$$u = s + t. \tag{2}$$

Example 1. Let's solve $x^3 - 15x - 4 = 0$. To make things easy on ourselves, this is already depressed. We note that $e = -15$ and $f = -4$. Then (2) becomes

$$\begin{aligned}
u &= s + t \\
&= -\frac{-15}{3\sqrt[3]{\frac{4}{2} \pm \frac{\sqrt{16-(-4)(-15)^3/27}}{2}}} + \sqrt[3]{\frac{4}{2} \pm \frac{\sqrt{16 - (-4)(-15)^3/27}}{2}} \\
&= \frac{5}{\sqrt[3]{2 \pm \sqrt{-121}}} + \sqrt[3]{2 \pm \sqrt{-121}} \\
&= \frac{5}{\sqrt[3]{2 \pm \sqrt{-121}}} \cdot \frac{\sqrt[3]{2 \mp \sqrt{-121}}}{\sqrt[3]{2 \mp \sqrt{-121}}} + \sqrt[3]{2 \pm \sqrt{-121}} \\
&= \frac{5\sqrt[3]{2 \mp \sqrt{-121}}}{\sqrt[3]{4 - (-121)}} + \sqrt[3]{2 \pm \sqrt{-121}} \\
&= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\
&= 4.
\end{aligned}$$

Remark 2. What about the other roots? I played a little fast and loose in (1) when I took the cube root of t^3. Given the primitive third root of unity $\omega = \frac{-1+i\sqrt{3}}{2}$, the cube roots of t^3 are $t, \omega t, \omega^2 t$.