Symbolic Powers of Edge Ideals

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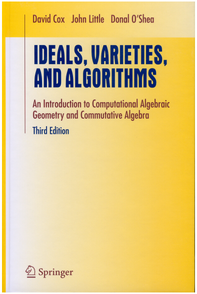
Our project

Background: a student approached me to do an honors contract in a special topics course.



My research area: commutative algebra/algebraic geometry

Our situation

Let k be an algebraically closed field (e.g., $k = \mathbb{C}$).

We will primarily consider homogeneous ideals $I \subseteq R = k[x_0, x_1, \ldots, x_N]$.
[The word *form* is interchangeable with *homogeneous polynomial*.]

Example

In $\mathbb{C}[X, Y, Z]$ such an ideal is $I = (XZ, YZ, X^3 - 3X^2Y - XY^2)$.
A non-example is $J = (X^2 - Y, Z^2)$.

Ordinary Powers

Given a ideals $I, J \subseteq R$, we may multiply ideals. Recall:

$$IJ = (FG \,:\, F \in I,\, G \in J).$$

We may extend this to (ordinary) powers:

$$I^r = (G_{i_1} G_{i_2} \cdots G_{i_r} \,:\, G_i \in I)$$

Example

Let $I = (X, Y) \subseteq \mathbb{C}[X, Y, Z]$. Then $I^2 = (X^2, XY, Y^2)$, $I^3 = (X^3, X^2Y, XY^2, Y^3)$, etc.

Note: We have $I^r \subseteq I^t$ if and only if $r \geq t$. *ideal gets (strictly) smaller*

Symbolic Powers

Definition

Given an ideal $I \subseteq R$, we define the m-th symbolic power of I to be

$$I^{(m)} = R \cap \left(\bigcap_P (I^m R_P) \right).$$

This can reduce to a much cleaner definition if more information about I is available.

Note: We have $I^{(r)} \subseteq I^{(t)}$ if and only if $r \geq t$.

Ordinary vs. Symbolic

Question

What is the relationship between I^r and $I^{(m)}$?

Answer: It depends on I.

A partial answer: $I^r \subseteq I^{(m)}$ if and only if $r \geq m$.

A (further) partial answer: $I^{(m)} \subseteq I^r$ implies $m \geq r$.

Before elaborating, we ask: what can symbolic powers look like?

Symbolic Powers of Edge Ideals

First studied by R. Villareal in the 1990s

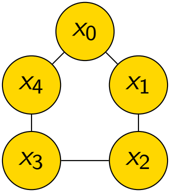
Let $V = \{x_1, x_2, \ldots, x_n\}$ be a set of variables and consider the (simple) graph $G = (V, E)$, where E contains 2-element sets comprised of pairs of the variables (so, e.g., $\{x_1, x_2\} \in E$ but $\{x_1, x_2, x_3\}, \{x_1^2\} \notin E$).

Definition

Given $G = (V, E)$ as above, the edge ideal of G is $I(G) = (x_i x_j : \{x_i, x_j\} \in E) \subseteq k[x_1, x_2, \ldots, x_n]$.

Fact: For an edge ideal I, $I^{(m)} = \bigcap_i P_i^m$, where the P_i correspond to minimal vertex covers of G.

$I = I(C_5) = (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_0)$



Here, the ring is $R = k[x_0, x_1, x_2, x_3, x_4]$, and the ideals corresponding to minimal vertex covers are $P_1 = (x_0, x_1, x_3)$, $P_2 = (x_0, x_2, x_3)$, $P_3 = (x_0, x_2, x_4)$, $P_4 = (x_1, x_2, x_4)$, $P_5 = (x_1, x_3, x_4)$. Then

$$\begin{aligned} I^{(2)} &= P_1^2 \cap P_2^2 \cap P_3^2 \cap P_4^2 \cap P_5^2 \\ &= (x_0^2x_1^2, x_0x_1^2x_2, x_1^2x_2^2, x_0x_1x_2x_3, x_1x_2^2x_3, x_2^2x_3^2, x_0^2x_1x_4, x_0x_1x_2x_4, \\ &\qquad x_0x_1x_3x_4, x_0x_2x_3x_4, x_1x_2x_3x_4, x_2x_3^2x_4, x_0^2x_4^2, x_0x_3x_4^2, x_3^2x_4^2) \\ &= I^2. \end{aligned}$$

But $I^{(t)} \neq I^t$ for all $t > 2$.

Bipartite edge ideal characterization

Theorem (Simis-Vasconcelos-Villareal (1994))

Given an edge ideal $I = I(G) \subseteq k[x_1, x_2, \ldots, x_n]$ as above, the following are equivalent.

(i) $I^{(m)} = I^m$ *for all* $m \geq 1$.

(ii) *The graph* G *is bipartite.*

Edge ideals of non-bipartite graphs

A consequence of the previous theorem is: if G is not bipartite and $I = I(G)$, then there exists a $t > 0$ such that $I^{(t)} \neq I^t$.

Our main question:

Problem

If $I = I(G)$ and G is not bipartite, how do $I^{(m)}$ and I^r compare?

Problem (Invariant Problem)

Compute invariants related to the containment $I^{(m)} \subseteq I^r$.

A conjecture

Focus of the honors project at Dordt College in Spring 2015: what happens when G is not bipartite?

Conjecture (Ellis–Wilson–McLoud-Mann)

Let $I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}]$ be the edge ideal of the odd cycle on $2n+1$ vertices. Then

- $I^t = I^{(t)}$ *for all* $1 \leq t \leq n$;
- $I^t \neq I^{(t)}$ *for all* $t > n$.

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Of importance when discussing ideal containments is the *initial degree*.

Definition

Let $J \subsetneq k[x_0, x_1, \ldots, x_N]$ *be a nonzero homogeneous ideal. Define*

$$\alpha(J) = \min\{d \ : \ \textit{there exists } 0 \neq f \in J, \deg(f) = d\}.$$

Note: if $\alpha(I^{(m)}) < \alpha(I^r)$ then $I^{(m)} \not\subseteq I^r$.

Example

Given an edge ideal $I = I(G)$, $\alpha(I) = 2$ and $\alpha(I^r) = r\alpha(I) = 2r$. Computing $\alpha(I^{(m)})$ is more delicate.

Given I, the edge ideal of C_{2n+1},

$$\alpha(I^{(m)}) = 2m - \lfloor \frac{m}{n+1} \rfloor$$

Half-proof of the conjecture

Proposition

Let $I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}]$ *be the edge ideal of the odd cycle on* $2n+1$ *vertices. Then* $I^{(t)} \neq I^t$ *for all* $t > n$.

Proof.

We know $\alpha(I^t) = 2t$ and $\alpha(I^{(t)}) = 2t - \lfloor \frac{t}{n+1} \rfloor \leq 2t - 1 < 2t$ when $t > n$. □

Our work attempting to prove the rest of the conjecture is ongoing.

Thanks

Thank you!